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LETTER TO THE EDITOR

The critical exponent γ for the three-dimensional Ising model

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Abstract. Estimates for the critical exponent γ for the initial susceptibility of the three-dimensional spin-$\frac{1}{2}$ Ising model are summarised. There is a discrepancy between estimates based on high-temperature series expansions and those obtained using renormalisation group theory. High-temperature series estimates for γ are reviewed and re-examined using some new data. It is tentatively concluded that a small discrepancy still appears to exist and that further work is needed to resolve it.

The precise value of the critical exponent γ for the initial susceptibility of the three-dimensional spin-$\frac{1}{2}$ Ising model (with nearest-neighbour isotropic interactions) is of great theoretical significance. Of particular interest is the discrepancy that has emerged between estimates of γ based on an analysis of high-temperature expansions for the susceptibility and those obtained using renormalisation group (RG) theory. It is this discrepancy that we review briefly in this letter.

Explicitly the exponent γ describes the divergence of the initial susceptibility χ_0 as the Curie temperature is approached from above:

$$\chi_0 \sim (T - T_c)^{-\gamma}, \quad T \to T_c+. \tag{1}$$

Until quite recently the most successful way of estimating γ was the method of exact series expansions (reviewed by Domb 1974) coupled with various extrapolation techniques (reviewed by Gaunt and Guttmann 1974). One of the earliest analyses of this kind appears to have been that of Domb and Sykes (1957) who initially suggested $\gamma = 1{\cdot}250$ for the simple cubic and face-centred cubic lattices, and $\gamma = 1{\cdot}244$ for the body-centred cubic lattice. Subsequently they made the conjecture that $\gamma = 1\frac{1}{4}$ for *all* three-dimensional lattices (Domb and Sykes 1961), and this proved to be an important influence in the formulation of the universality hypothesis (Kadanoff 1971, Griffiths 1970). Over the years, as series were extended and extrapolation techniques refined, numerical evidence steadily accumulated supporting $\gamma = 1\frac{1}{4}$ with confidence limits of the order of 10^{-3}. As a typical estimate we quote from the well known review article of Fisher (1967)

$$\gamma = 1{\cdot}250 \pm 0{\cdot}003. \tag{2}$$

A few years ago, Wilson (1971a, b) initiated the RG approach to critical phenomena. This has greatly increased our understanding of phase transitions and has, for example, put the universality hypothesis on a firm theoretical basis. Critical exponents have been calculated through the $\epsilon = 4 - d$ expansion (Wilson and Fisher 1972), and more recently use has been made of perturbation series for the $g\phi^4$ field theory directly in three dimensions (Baker *et al* 1976). Using Padé–Borel summation techniques they obtained

$$\gamma = 1{\cdot}241 \pm 0{\cdot}002. \tag{3}$$

More extensive work (Baker *et al* 1978), incorporating the known asymptotic behaviour of the coefficients in the perturbation series (Brézin *et al* 1977), gave the result

$$\gamma = 1{\cdot}241 \pm 0{\cdot}004. \tag{4}$$

By making certain analyticity assumptions, these estimates have been further refined by Le Guillou and Zinn-Justin (1977) who give

$$\gamma = 1{\cdot}2402 \pm 0{\cdot}0009. \tag{5}$$

The RG estimates (3) to (5) are definitely lower than the central high-temperature series estimate (2). The difference is only $\frac{4}{5}$% but is too large to be explained away convincingly by the quoted confidence limits. Furthermore it would appear that γ can now be measured experimentally to this degree of precision (Chang *et al* 1976, Hocken and Moldover 1976). Clearly the resolution of this discrepancy is a matter of great theoretical and experimental importance.

In an attempt to shed some light on this problem we have returned to take another close and, we hope, unbiased look at the series data. It should be noted that some extensive recent studies (for example those of Sykes *et al* 1972) have been aimed primarily at providing estimates of other parameters (notably critical temperatures and amplitudes) on the assumption that γ is exactly $1\frac{1}{4}$. The discrepancy with which we are concerned is probably too small to affect the outcome of such investigations significantly. We have now to re-examine the data with a new emphasis.

The series are in the usual high-temperature variable $v = \tanh K$ and are published elsewhere through v^N, where $N = 15$ for the face-centred cubic (FCC) and body-centred cubic (BCC) lattices (McKenzie 1975, Sykes *et al* 1972) and $N = 22$ for the diamond (D) lattice (Gaunt and Sykes 1973). For the simple cubic lattice we have corrected an insignificant error in the last term ($N = 17$) given by Sykes *et al* (1972) and added two new coefficients giving

$$\chi_0 = 1 + \ldots + 401\,225\,368\,086 v^{17} + 1864\,308\,847\,838 v^{18} + 8660\,961\,643\,254 v^{19} + \ldots . \tag{6}$$

We have taken elaborate precautions to ensure the accuracy of these coefficients, and details of the derivation will be published elsewhere. Some of the methods used are described by Sykes (1979) for the analogous problem to $N = 17$ for the four-dimensional simple hypercubic lattice.

The analysis that we present below involves only minor refinements of the 'classic' ratio method used by Domb and Sykes (1957) in their pioneering study of this problem. However, rather than work with expansions in powers of v, we have preferred first to transform to a new variable x defined by

$$x = 2v/(1 + v/v_c^*) \tag{7}$$

where v_c^* is a good estimate of the exact critical point at $v = v_c$. We have used the estimates of v_c given by Sykes *et al* (1972) and Gaunt and Sykes (1973). The transformation (7) maps the point $v = -v_c^*$ to infinity, while the point $v = v_c^*$ is a fixed point (as is the origin $v = 0$). This means that for the loose-packed lattices the antiferromagnetic singularity at $v=-v_c \simeq -v_c^*$ is mapped so far from the origin relative to the ferromagnetic singularity that it should not affect the extrapolations significantly.

The exponent γ may be obtained by extrapolating to $n=\infty$ the successive estimates

$$1+n(\mu_n/\mu-1) \tag{8}$$

where $\mu_n = a_n/a_{n-1}$ is the ratio of successive coefficients of the x expansion and $\mu = 1/x_c$. Since x_c is not known exactly we avoid the necessity of using the estimate $x_c \simeq v_c^*$ by simply replacing μ in (8) by an nth-order estimate μ_n^* which is known to approach μ as $n\to\infty$, namely

$$\mu_n^* = n\mu_n - (n-1)\mu_{n-1} \tag{9}$$

corresponding to extrapolating successive ratios linearly against $1/n$. Hence (8) becomes

$$1+n(\mu_n/\mu_n^*-1)=\gamma_n^* \tag{10}$$

from which we obtain our final estimates γ_n of γ by linear extrapolation against $1/n$. The estimates are plotted against $1/n$ in figure 1. These estimates are 'unbiased' in the sense that they do not depend directly on an estimate of the critical point x_c of the transformed series. It is true that we have used v_c^* in calculating the transformed series $\chi_0(x)$, but this was simply to reduce the effect of the antiferromagnetic singularity; this could have been achieved quite well with a much less accurate value of v_c^*. In fact the estimates γ_n turn out to be very stable with respect to small changes in v_c^*; for example

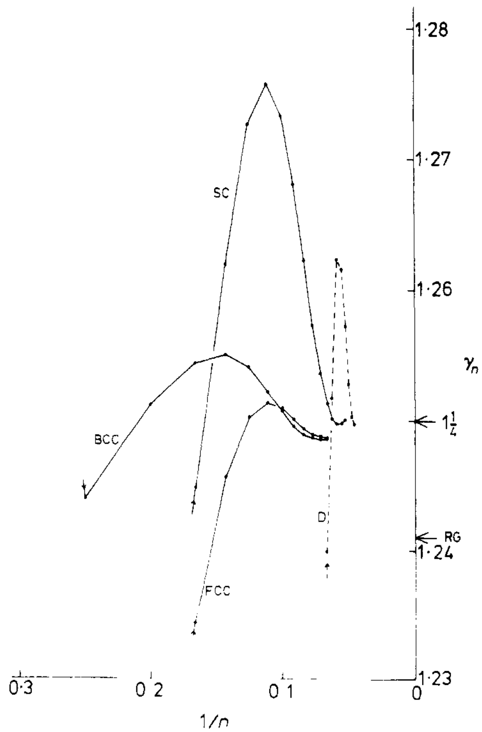

Figure 1. Successive estimates γ_n for γ, versus $1/n$. Changes in the critical temperature used to transform from v to x do not affect this graph for any lattice if Δv_c is within quoted confidence limits. The arrows indicate the limit $1\frac{1}{4}$ (Domb and Sykes 1961) and the RG limit 1·241 (Baker *et al* 1978).

the uncertainties in v_c quoted by Sykes *et al* (1972) and Gaunt and Sykes (1973) do not affect the last estimates of γ until at least the sixth decimal place.

An examination of figure 1 suggests that apart from the diamond lattice (which has probably not yet been taken quite far enough) the successive approximations are becoming relatively smooth and slowly varying (in this connection the large scale used for the vertical axis should be noted). The sequences are all in good accord with the view that they have a common limit; on this assumption an objective choice would seem to lie within the confidence limits of the estimate (1) which we have arrowed.

The way our plots approach γ could be affected by higher-order confluent singularities. We have tried to take these into account by fitting in various ways to the asymptotic behaviour indicated by RG theory, but so far without notable success. A very thorough analysis of $\chi_0(v)$ was performed along these lines by Camp and Van Dyke (1975) who concluded that the amplitudes of the leading corrections are either very small or vanish identically. Convergence could also be marred by the presence of non-confluent singularities; our transformation of variables should at least have reduced the effect of the antiferromagnetic singularity.

We are impressed by the essential consistency of the results for all four lattices illustrated in figure 1 which seem to support the appealingly simple hypothesis that γ is exactly $1\frac{1}{4}$. Further expansion coefficients would evidently be of great interest, and we are currently investigating the possibility of obtaining some. Of course the evidence we have presented does not disprove the renormalisation group theory predictions; rather we would say it provides a stimulus for further research, since if γ is close to 1·241 it remains to be explained why it is so difficult to detect this from series expansions.

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